VIBRATION ENGINEERING No.1

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Vibration mechanics is very important for engineering, particular to construction engineering. Especially now, architecture and civile engineers have to study vibration of a construction in which human being, when we consider "Hanshin Dai Shinsai" in which more than 5,000 people dead. I studied vibration mechanics, in civil engineering department, in graduate school. Inventions relating to a earthquake proof construction, control vibration and the like might be invented after the earthquake, and be filled, so that I explain vibration mechanics few times on "PATENT". Hereinafter I will describe BASIC PROBLEM in this paper.

1. Single Degree Of Freedom System

Consider Single Degree Of Freedom System ("S.D.O.F.") as shown in Fig. 1.

(1) Equation Of Motion

$f(t) = P \cdot Re(e^{i\omega t})$ f(t) acting to subject is

(1)

Due to D'Alembert's principle, equation of motion is

$$m\ddot{x} + c\dot{x} + kx = P \cdot Re(e^{i\omega t})$$

$$\ddot{x} + 2\beta\omega_0\dot{x} + \omega_0^2 x = \frac{P}{m} \cdot Re(e^{i\omega t})$$

ω_0 : natural circular frequency

β : damping constant

$$\dot{x} = \frac{\partial x}{\partial t}, \quad \ddot{x} = \frac{\partial^2 x}{\partial t^2}$$

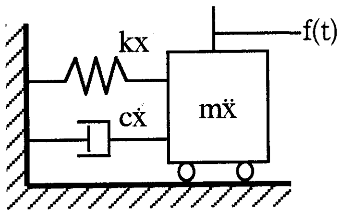

Fig. 1

(2) Response

General solution x

$$x = x_h + x_p$$

x_h : homogenious solution, x_p : particular solution

$$x_h = e^{-\beta\omega_0 t}(a_1 \cos\omega_d t + a_2 \sin\omega_d t), \qquad \omega_d = \sqrt{1-\beta^2}\,\omega_0 \quad \text{in case of } \beta \le 1.$$

Assume x_p is

$$x_p(t) = Ae^{i\omega t}$$

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substituting x_p into equa. (1)

$$A=\frac{1}{m(\omega_0^2-\omega^2+2i\beta\omega_0\omega)}\equiv H(i\omega)$$

$H(i\omega)$: complex frequency response function

$$A=\left|H(i\omega)\right|e^{i\varphi(\omega)}$$

where $\varphi(\omega)= -\mathrm{Arg}H(i\omega)$, which is phase shift.

$$x_p(t)=\left|H(i\omega)\right|e^{i(\omega t-\varphi(\omega))}=\frac{1}{m\sqrt{\left\{(\omega_0^2-\omega^2)^2+(2i\beta\omega_0\omega)^2\right\}}}e^{i(\omega t-\varphi(\omega))}$$

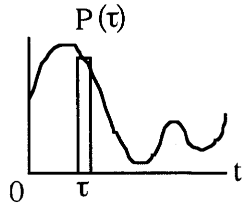
$$\therefore\ x=e^{-\beta\omega_0 t}(a_1\cos\omega_d t+a_2\sin\omega_d t)+\frac{1}{m\sqrt{\left\{(\omega_0^2-\omega^2)^2+(2i\beta\omega_0\omega)^2\right\}}}e^{i(\omega t-\varphi(\omega))}$$

<u>Response To Transient Force</u>

Above external force is periodic force. In case of transient force, x is indicated as follows, which is called to "unit response function".

$$x=\frac{1}{m\omega_d}e^{-\beta\omega_0 t}\sin\omega_d t$$

$$x_p=\frac{1}{m\omega_d}\int_0^t P(\tau)e^{-\beta\omega_0(t-\tau)}\sin\omega_d(t-\tau)d\tau$$



$$\therefore\ x=e^{-\beta\omega_0 t}(a_1\cos\omega_d t+a_2\sin\omega_d t)+\frac{1}{m\omega_d}\int_0^t P(\tau)e^{-\beta\omega_0(t-\tau)}\sin\omega_d(t-\tau)d\tau$$

2. Two Degrees Of Freedom System

Next, I will describe how to derive vibration equation of Two Degrees Of Freedom System (hereinafter called "T.D.O.F.") due to Lagrangean Equation.

(1) Lagrangean Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)-\frac{\partial L}{\partial x_i}+\frac{\partial F}{\partial \dot{x}_i}-Q_i=0 \qquad i=1,2,3,4,5\cdots,\cdots$$

$i=1,2$ when two – degree – of – freedom system

$L = T - V$

T : kintic energy, V : potencial energy, F : dissipation, Q : genealized energy,

here $u = u(\eta, t) = \psi(\eta) \cdot x(t)$

(2) Rayleigh-Ritz approach to tower-equipment system

I will consider a model as shown in Fig. 2. It is assumed that the tower has mainly bending, and the observation deck (η=L) has mainly rigid-body translation. The equipment at η=L is modelled single-degree of freedom oscillator with parameters m, c, k.

Assuming that the vibration shape of the tower is

$$\psi(\eta) = \frac{1}{11}\left\{20(\frac{\eta}{L})^2 - 10(\frac{\eta}{L})^3 + (\frac{\eta}{L})^5\right\}$$

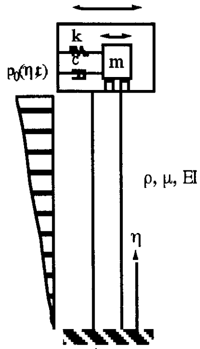


Fig. 2

and defining x_1 is the absolute displacement of the top of the tower and x_2 is the relative displacement of the relative to the deck of the top of the tower. Then the following equation of motion may be derived.

when x_1 is absolute displacement of the tower and

x_2 is relative displacement of the deck

$$T = \underbrace{\frac{1}{2}\left(\frac{21{,}128}{83{,}853}\rho L + M\right)\dot{x}_1^2}_{\text{tower \& deck}} + \underbrace{\frac{1}{2}m\left(\dot{x}_1 + x\dot{x}_2\right)^2}_{\text{equipment}}$$

$$V = \underbrace{\frac{1}{2}\left(\frac{2{,}640}{847}\frac{EI}{L^3}\right)x_1^2}_{\text{tower}} + \underbrace{\frac{1}{2}kx_2^2}_{\text{equipment}}$$

$$Q = \frac{2}{7}Lp_0(t) - \left(\frac{21{,}128}{83{,}853}\nu L\right)x\dot{x}_1$$

$\because \delta_2 = 0$, thererfore no viscous force

due to dashpot of equipment

$F = -c\dot{x}_2$

$$\begin{pmatrix} \frac{21,128}{83,853}\rho L + M + m, & m \\ m, & m \end{pmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{pmatrix} \frac{21,128}{83,853}\nu L, & 0 \\ 0, & 0c \end{pmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} \frac{240}{77}\frac{EI}{L^3}, & 0 \\ 0, & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} \frac{2}{7} L p_0(t) \\ 0 \end{Bmatrix}$$

$$\Downarrow$$

$$\boldsymbol{M\ddot{X}} + \boldsymbol{C\dot{X}} + \boldsymbol{KX} = \boldsymbol{F}$$

$\boldsymbol{M}$: mass matrix, $\boldsymbol{C}$: damping matrix, $\boldsymbol{K}$: stiffness matrix, $\boldsymbol{F}$: force matrix

In the above, T.D.O.F was solved, but the other model so-called "M.D.O.F (Many Degrees Of Freedom System)" can be of course solved due to the same method,
n × n matrix as follow may be derived.

$$\begin{pmatrix} \ddots & & & \\ & \ddots & & \\ & & m & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \ddot{x} \\ \vdots \end{pmatrix} + \begin{pmatrix} \ddots & & & \\ & \ddots & & \\ & & c & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \dot{x} \\ \vdots \end{pmatrix} + \begin{pmatrix} \ddots & & & \\ & \ddots & & \\ & & k & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ x \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ f(t) \\ \vdots \end{pmatrix}$$

(3) Modal Analysis

It is possible to solve above vibration equation due to "Modal Analysis".

$$\boldsymbol{M\ddot{X}} + \boldsymbol{C\dot{X}} + \boldsymbol{KX} = \boldsymbol{f}(t) \quad (1)$$

The first, excepting damping matrix and force matrix from above matrix equation.

$$\boldsymbol{M\ddot{X}} + \boldsymbol{KX} = 0 \quad (2)$$

Assuming solution of Eq. (2)

$$\boldsymbol{X} = \phi e^{i\omega t} \quad (3)$$

Substituting Eq. (3) into (2)

$$\left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right)\boldsymbol{\phi} = 0 \quad (4)$$

$\boldsymbol{\phi} \neq 0$ in order that significant solution exists,
hence coefficient matrix of $\boldsymbol{\phi}$ is 0,
therefore,

$$\det\left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right) = 0 \tag{5}$$

ω satisfying Eq. (5) is natural circular frequency ("eigenvalue" in mathematic field), ϕ corresponding to ω is natural frequency mode ("enigen vector"in mathematic field).

ω of which number is (n) is derived by solving Eq. (5) in n-degrees of freedom system,

Each ω is different from the others.

> $\omega_1 < \omega_2 < \omega_3 < \quad < \omega_n$
>
> ω_1: 1st natural frequency
>
> ω_2: 2nd
>
> ⋮
>
> ω_n: nth

substituting solution ω_i into Eq. (2),

$$\boldsymbol{K\phi} - \omega_i^2 \boldsymbol{M\phi} = 0 \qquad i = 1,2,3,\cdots,n \tag{6}$$

hereby

$$\boldsymbol{\phi} = \boldsymbol{\phi}_i \qquad \boldsymbol{i} = 1,2,3,\cdots,n$$

according to orthogonality,

$$\boldsymbol{\phi}_j^T \boldsymbol{M} \boldsymbol{\phi}_i = 0 \quad (i \neq j) \tag{7}$$

Solution of Eq. (1) is found as follows.

The first, assuming that solution of Eq. (1) is

$$\boldsymbol{X} = \boldsymbol{\Phi\eta}(\mathrm{t}) \tag{8}$$

$\boldsymbol{\Phi}$ is modal matrix, which is given by following equation.

$$\boldsymbol{\Phi} = [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \cdots, \boldsymbol{\phi}_n] = \begin{bmatrix} \phi_{11}, & \phi_{12}, & \cdots, & \phi_{1n} \\ \phi_{21}, & \phi_{22}, & \cdots, & \phi_{2n} \\ \cdots, & \cdots, & \cdots, & \cdots \\ \phi_{n1}, & \phi_{n2}, & \cdots, & \phi_{nn} \end{bmatrix} : \mathrm{n} \times \mathrm{n\ matrix} \tag{9}$$

Each line of matrix $\boldsymbol{\Phi}$ is namely constituted as each natural vibration mode.

Substituting Eq. (8) into (1)

$$\boldsymbol{M\Phi\ddot{\eta}} + \boldsymbol{K\,\Phi\eta} = \boldsymbol{F} \tag{10}$$

$$\boldsymbol{\Phi}^T\boldsymbol{M\Phi\ddot{\eta}} + \boldsymbol{\Phi}^T\boldsymbol{K\,\Phi\eta} = \boldsymbol{\Phi}^T\boldsymbol{F} \tag{11}$$

$$\boldsymbol{\Phi}^T\boldsymbol{M\Phi} = [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \cdots, \boldsymbol{\phi}_n]^T \boldsymbol{M} [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \cdots, \boldsymbol{\phi}_n] = \begin{bmatrix} \boldsymbol{\phi}_1^T\boldsymbol{M\phi}_1, & \boldsymbol{\phi}_1^T\boldsymbol{M\phi}_2, & \cdots, & \boldsymbol{\phi}_1^T\boldsymbol{M\phi}_n \\ \boldsymbol{\phi}_2^T\boldsymbol{M\phi}_1, & \boldsymbol{\phi}_2^T\boldsymbol{M\phi}_2, & \cdots, & \boldsymbol{\phi}_2^T\boldsymbol{M\phi}_n \\ \cdots, & \cdots, & \cdots, & \cdots \\ \boldsymbol{\phi}_n^T\boldsymbol{M\phi}_1, & \boldsymbol{\phi}_n^T\boldsymbol{M\phi}_2, & \cdots, & \boldsymbol{\phi}_n^T\boldsymbol{M\phi}_n \end{bmatrix} \tag{12}$$

due to orthogonality, all of nonsymetric elements is 0, hence

$$\boldsymbol{\Phi}^T \boldsymbol{M} \boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1^T \boldsymbol{M} \boldsymbol{\phi}_1, & & & \\ & \boldsymbol{\phi}_2^T \boldsymbol{M} \boldsymbol{\phi}_2, & & \\ & & \ddots & \\ & & & \boldsymbol{\phi}_n^T \boldsymbol{M} \boldsymbol{\phi}_n, \end{bmatrix} \quad (13)$$

similarly,

$$\boldsymbol{\Phi}^T \boldsymbol{K} \boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1^T \boldsymbol{K} \boldsymbol{\phi}_1, & & & \\ & \boldsymbol{\phi}_2^T \boldsymbol{K} \boldsymbol{\phi}_2, & & \\ & & \ddots & \\ & & & \boldsymbol{\phi}_n^T \boldsymbol{K} \boldsymbol{\phi}_n, \end{bmatrix} \quad (14)$$

since

$$\phi_i^T \boldsymbol{K} \phi_i = \omega_i^2 \phi_i^T \boldsymbol{M} \phi_i \, , \quad (15)$$

$$\boldsymbol{\Phi}^T \boldsymbol{K} \boldsymbol{\Phi} = \begin{bmatrix} \omega_1^2 \boldsymbol{\phi}_1^T \boldsymbol{M} \boldsymbol{\phi}_1, & & & \\ & \omega_2^2 \boldsymbol{\phi}_2^T \boldsymbol{M} \boldsymbol{\phi}_2, & & \\ & & \ddots & \\ & & & \omega^2{}_1 \boldsymbol{\phi}_n^T \boldsymbol{M} \boldsymbol{\phi}_n, \end{bmatrix} \quad (16)$$

therefore

$$\begin{bmatrix} \boldsymbol{\phi}_1^T \boldsymbol{M} \boldsymbol{\phi}_1, & & & \\ & \boldsymbol{\phi}_2^T \boldsymbol{M} \boldsymbol{\phi}_2, & & \\ & & \ddots & \\ & & & \boldsymbol{\phi}_n^T \boldsymbol{M} \boldsymbol{\phi}_n, \end{bmatrix} \ddot{\boldsymbol{\eta}} + \begin{bmatrix} \omega_1^2 \boldsymbol{\phi}_1^T \boldsymbol{M} \boldsymbol{\phi}_1, & & & \\ & \omega_2^2 \boldsymbol{\phi}_2^T \boldsymbol{M} \boldsymbol{\phi}_2, & & \\ & & \ddots & \\ & & & \omega_{n1}^2 \boldsymbol{\phi}_n^T \boldsymbol{M} \boldsymbol{\phi}_n, \end{bmatrix} \boldsymbol{\eta}$$

$$= \boldsymbol{\Phi}^T \boldsymbol{F} \quad (17)$$

Eq. (21) means that quadratic differential Eq. (10) is transformed to non-coupled equation. This is the most characteristic of Modal Analysis.

Solving differential equation

$$\phi_i^T \boldsymbol{M} \phi_i \eta_i + \omega_i^2 \phi_i^T \boldsymbol{M} \phi_i \eta_i = \phi_i^T F \quad i = 1,2,3, \cdots, \mathrm{n} \quad (18)$$

substituting solution η_i into Eq. (15), solution $\boldsymbol{X}$ can be found.

In case in which damping matrix is considered, if damping matrix is symmetric as follows,

$$\boldsymbol{\Phi}^T \boldsymbol{C} \boldsymbol{\Phi} = \mathrm{diag}\left(2\beta_1\omega_1, 2\beta_2\omega_2, \cdots, 2\beta_n\omega_n\right) = \begin{pmatrix} 2\beta_1\omega_1 & & & \\ & 2\beta_2\omega_2 & & \\ & & \ddots & \\ & & & 2\beta_n\omega_n \end{pmatrix} \quad (19)$$

Modal Analysis is applicable. In actual analyzing, β_i is assumed so that Eq. (19) may be formed.

Modal Analysis is, as above, theoretically systematic, in which solution can be simultaneously found while considering vibration property, but linear-M.D.O.F. of non-coupled system can be only applied.

When $\boldsymbol{C}\boldsymbol{M}^{-1}\boldsymbol{K} = \boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{C}$ (proportional damping), the eigenvalue problem of Eq. (20) may be solved indirectly as follows:

$$(\lambda^2 \boldsymbol{M} + \lambda \boldsymbol{C} + \boldsymbol{K})\boldsymbol{a} = \boldsymbol{0}$$

(substituting $\boldsymbol{X} = \boldsymbol{a}e^{\lambda t}$ into (1))

$$\boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{M} \boldsymbol{M}^{-\frac{1}{2}} \equiv \boldsymbol{I} = \mathrm{diag}(1, \cdots, 1) = \begin{pmatrix} 1 & & & \\ & \ddots & \boldsymbol{0} & \\ & & \ddots & \\ & \boldsymbol{0} & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$\boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{K} \boldsymbol{M}^{-\frac{1}{2}} \equiv \overline{\overline{\boldsymbol{A}}}$$

$$(\mu^2 \boldsymbol{I} + \overline{\overline{\boldsymbol{A}}})\boldsymbol{b} = \boldsymbol{0}, \quad (\boldsymbol{b}^T \boldsymbol{b} = \boldsymbol{I})$$

for example, in case of T.D.O.F,

$$\omega_0^2 \begin{pmatrix} \mu^2 + a_{11}, & a_{12} \\ a_{21}, & \mu^2 + a_{22} \end{pmatrix} \boldsymbol{b} = \boldsymbol{0}, \quad \det \begin{pmatrix} \mu^2 + a_{11}, & a_{12} \\ a_{21}, & \mu^2 + a_{22} \end{pmatrix} = \boldsymbol{0}$$

$-\mu^2 =$ m or n, sustituting m,n thereinto

$$\Rightarrow \quad \omega_0^2 \begin{pmatrix} \cdot, & \cdot \\ \cdot, & \cdot \end{pmatrix} \boldsymbol{b}_1 = 0,\ \omega_0^2 \begin{pmatrix} \cdot, & \cdot \\ \cdot, & \cdot \end{pmatrix} \boldsymbol{b}_2 = 0 \quad \Rightarrow \quad \boldsymbol{b}_1 = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix},\ \boldsymbol{b}_2 = \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} \quad (\rightarrow \boldsymbol{B} = (\boldsymbol{b}_1, \boldsymbol{b}_2))$$

$$\omega_1 = \sqrt{-\mu_1^2}\,\omega_0,\ \omega_2 = \sqrt{-\mu_2^2}\,\omega_0$$

$$\Downarrow$$

$$\boldsymbol{B}^{\mathrm{T}} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{K} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{B} = \begin{pmatrix} -\mu_1^2, & 0 \\ 0, & -\mu_2^2 \end{pmatrix},\quad \boldsymbol{B}^{\mathrm{T}} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{C} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{B} = \begin{pmatrix} \tau_1^2, & 0 \\ 0, & \tau_2^2 \end{pmatrix}$$

Substituting $\boldsymbol{X} = \boldsymbol{B}\boldsymbol{\eta}$

$$\boldsymbol{B}^{\mathrm{T}} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{M} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{B} \ddot{\boldsymbol{\eta}} + \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{C} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{B} \dot{\boldsymbol{\eta}} + \boldsymbol{B}^{\mathrm{T}} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{K} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{B} \boldsymbol{\eta} = \boldsymbol{B}^{\mathrm{T}} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{f} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{B}$$

$$\therefore \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ddot{\boldsymbol{\eta}} + \begin{pmatrix} \tau_1^2 & 0 \\ 0 & \tau_2^2 \end{pmatrix} \dot{\boldsymbol{\eta}} + \begin{pmatrix} -\mu_1^2 & 0 \\ 0 & -\mu_2^2 \end{pmatrix} \boldsymbol{\eta} = \boldsymbol{B}^{\mathrm{T}} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{f} \boldsymbol{M}^{-\frac{1}{2}} \boldsymbol{B}$$

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